A two-factor linguistic time series model with time-dependent semantic relationship groups

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ABSTRACT

Fuzzy time series-based forecasting models have been extensively examined because of their flexibility and ability to handle uncertainty. In those forecasting models, the universe of discourse is apportioned into subintervals and human experts intuitively assign fuzzy sets to them. Because fuzzy sets are always associated with linguistic words, the concept of linguistic time series (LTS) is introduced, in which a numeric time series is transformed into a linguistic one by a mathematical formalism. Then, the obtained LTS specifies semantic logical relationships that are utilized to generate semantic logical relationship groups (SLRG) for establishing forecasting models. In this paper, the concept of time-dependent multi-factor SLRG is proposed, and it is used to establish time-dependent multi-factor first-order and high-order LTS models. The experimental studies executed on the time series datasets of the TAIFEX index of Taiwan and daily average temperature in 1996 in Taipei, Taiwan, show that our proposed forecasting model outperforms the benchmarked models.

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1. Introduction

In 1993, fuzzy time series-based forecasting models (FTS-FM) was first given introduction by Song and Chissom [1]. This forecasting model aims to utilize the theory of fuzzy set (FS) to solve the uncertainty of time series (TS) data. In 1996, Chen enhanced it by applying a simple arithmetic calculation method to reduce computational time [2]. Since then, it has been examined by the research community to enhance the methodology to improve the forecasted accuracy. The methodology enhancements focused on optimizing interval lengths by genetic algorithm (GA) [3], [4], particle swarm optimization (PSO) [5], [6], clustering [7], [8]; utilizing the weighted FLRs [9]; utilizing the high-order FLRs (HO-FLRs) [10], [11]; utilizing the multi-factor FLRs [11], [12]; enhancing the method of calculating crisp forecasting values [6]; using more complex FS types such as intuitive FSs [13], type-2 FSs [14], [15], picture FSs [16], hesitant FSs [17].

Because FSs are computational objects of the associated linguistic words, there is a need for a formal linkage bridge between FSs and words' qualitative semantics so that the word's semantics are preserved when operating on their computational semantic objects. Fortunately, hedge algebras (HAs) [18], [19] can be used to formalize and quantify the words' qualitative semantics of a linguistic variable (LV) in an axiomatic way. Therefore, it can be utilized to establish a mathematical basis for such formal linkage bridge. HAs have been modeled and applied to handle many practical problems, such as data mining

[20], image processing [21], fuzzy control [22], linguistic summarization [23], [24], forecasting [25], [26], [27], decision making [28], [29], and network security [30].

To efficiently solve real-world forecasting problems, Hieu et al. introduced the concept of linguistic time series (LTS) in 2020 by using HAs, and established a forecasting model based on LTS, so-called the LTS forecasting model (LTS-FM) [25]. In this LTS-FM, the universe of discourse (UoD) of the LV is directly described by the words generated by an associated HA. Next, the historical numeric TS is transformed into LTS by the numerical words' semantics. Then, the semantic logical relationships (SLRs) are generated, and based on these, the semantic logical relationship groups (SLRGs) are established. Finally, forecast and compute crisp forecasted values. However, the future SLTs are also used to forecast the past forecasted values in the training phase. This means that the SLTs that occur after the current time t are also included in the right-hand side (RHS) of the SLTGs under consideration. It sounds illogical in the context of solving a machine learning problem. This issue was first addressed by Dieu [31] for FTS-FM in 2014 in such a way that only FSs, which occur no earlier than t, are included on the RHS of a fuzzy logical relationship group (FLRG). The established FLRGs are so-called time-dependent FLRGs, and they are applied to efficiently solve real-world forecasting problems [32], [33]. This mechanism was utilized for establishing time-dependent SLRGs for the single-factor LTS-FM by Phong in 2021 [26].

This paper proposes the concepts of multi-factor first-order and high-order time-dependent SLRGs inspired by the time-dependent SLRGs proposed in [26]. Based on those, the multi-factor first-order and high-order time-dependent LTS-FMs are established. In our models, the time-dependent SLTGs are used instead of the time-independent SLTGs in the existing LTS-FMs [27], [34]. The experimental results executed on two TS datasets of the TAIFEX index of Taiwan and daily average temperature in 1996 in Taipei, Taiwan, show that our models produce higher forecast accuracies than the compared forecasting models.

The rest of this paper is as follows: Section 2 gives a brief presentation of the background knowledge such as HA, LTS, FTS-FM, and PSO. Then, the section delves into the proposed concept of time-dependent two-factor SLTG and its application to the design of multi-factor LTS-FMs. Section 3 presents the evaluations of our models versus the benchmarked ones. Conclusions are given in Section 4.

2. Method

2.1. Theoretical background

In this subsection, the background of related theories is summarized, such as the theory of HAs, LTS, and PSO. That background knowledge is the basis for our proposed forecasting models.

2.2. Hedge algebras theory

Given introduction in 1990 by Ho and Wechler, hedge algebras (HAs) [19] is a mathematical framework that models LVs and their semantics using algebraic structures, enabling precise reasoning with vague or qualitative linguistic values. An HA of an LV $\mathcal{N}$ is the structure $\mathcal{A} = (\mathcal{W}, \mathcal{P}, \mathcal{G}, \mathcal{C}, \leq)$, where $\mathcal{W} \subseteq Dom(\mathcal{N})$ is a word set in the linguistic domain $Dom(\mathcal{N})$ of $\mathcal{N}$; $\mathcal{C} = \{0, W, 1\}$ includes a set of three word constants, where 0 is the smallest constant, W is the neutral constant, and 1 the greatest constant; $\mathcal{P} = \{p^-, p^+\}$ is the primary word set satisfying $p^- \leq p^+$; $\mathcal{G} = \mathcal{G}^- \cup \mathcal{G}^+$ is the set of linguistic hedges (LHs), where $\mathcal{G}^-$ is the negative hedge set and $\mathcal{G}^+$ is the positive hedge set; $\leq$ denotes the semantic order relations of the words in $Dom(\mathcal{N})$.

The linguistic words in $\mathcal{W}$ can be represented canonically as strings. Therefore, each word $y \in \mathcal{W}$ is represented as ωp, where $p \in \{p^-, p^+\}$ and ω is empty or $\omega = h_n \dots h_1, h_i \in \mathcal{G}, i = 1, \dots, n$. If $\mathcal{G}$ and $\mathcal{W}$ are linearly ordered, $\mathcal{A}$ is the linear hedge algebra. As we see that $\mathcal{G}(y) = \{\omega y, \omega \in \mathcal{G}\}$ includes all words which are induced from y by linearly acting all LHs in $\mathcal{G}$ on y. The semantics of each word in $\mathcal{G}(y)$ is changed by the impact of the LHs in $\mathcal{G}$, but it still conveys the original y's semantics.

Consequently, $\mathcal{G}(y)$ is considered the y's fuzziness, and the $\mathcal{G}(y)$'s diameter is the fuzziness measure (FM) of y, denoted by $\mathfrak{w}(y)$.

Definition 1. Given $\mathcal{A}$ a linear HA. A function $\mathfrak{w}: \mathcal{W} \to [0,1]$ is called the FM of the words in $\mathcal{W}$ if it satisfies the properties as follows [35]:

(P1): $\mathfrak{w}(p^-) + \mathfrak{w}(p^+) = 1$ and $\sum_{h \in \mathcal{G}} \mathfrak{w}(hw) = \mathfrak{w}(w)$, for $\forall w \in \mathcal{W}$;

(P2): $\mathfrak{w}(y) = 0$ for $\mathcal{G}(y) = y$. For word constants, we have $\mathfrak{w}(0) = \mathfrak{w}(W) = \mathfrak{w}(1) = 0$;

(P3): $\forall y, z \in \mathcal{W}, \forall h \in \mathcal{G}$, the ratio $\frac{\mathfrak{w}(hy)}{\mathfrak{w}(y)} = \frac{\mathfrak{w}(hz)}{\mathfrak{w}(z)}$ depends only on h, so it is called the FM of h and denoted by $\mu(h)$.

From (P1) and (P3), $\mathfrak{w}(y)$ represents the FM of y, where $y = h_n \dots h_1 p$, $p \in \{p^-, p^+\}$, is computed by $\mathfrak{w}(y) = \mu(h_n) \times \dots \times \mu(h_1) \times \mathfrak{w}(p)$, $\sum_{h \in \mathcal{G}} \mu(h) = 1$.

Definition 2: Semantically quantifying mapping (SQM) [35] of $\mathcal{A}$ is the mapping $\sigma: \mathcal{W} \to [0,1]$ provided that:

(S1): σ preserves the ordered structure on $\mathcal{W}$, i.e., $w \le z \to \sigma(w) \le \sigma(z), \forall w, z \in \mathcal{W}$;

(S2): σ is a one-to-one mapping, and the image of $\mathcal{W}$, $\sigma(\mathcal{W})$, is dense in [0, 1].

Proposition 1: Let $\mathfrak{w}$ be the FM in $\mathcal{W}$, $\sum_{i=-s}^{-1} \mu(h_i) = \gamma$, $\sum_{i=1}^{t} \mu(h_i) = \delta$, where $\gamma, \delta > 0$ and $\gamma + \delta = 1$. $\sigma(w)$ is calculated by $\mathfrak{w}$ as follows:

1) $\sigma(W) = \rho = \mathfrak{w}(p^-)$, $\sigma(p^-) = \rho - \gamma \mathfrak{w}(p^-) = \delta \mathfrak{w}(p^-)$, $\sigma(p^+) = \rho + \gamma \mathfrak{w}(p^+)$;
2) $\sigma(h_j w) = \sigma(w) + \text{sign}(h_j w)(\sum_{i=\text{sign}(j)}^{j} \mathfrak{w}(h_i w) - \varphi(h_j w)\mathfrak{w}(h_j w))$, where $-s \le j \le t \,\&\, j \ne 0\}$ and $\varphi(h_j w) = \frac{1}{2}[1 + \text{sign}(h_j w)\text{sign}(h_t h_j w)(\delta - \gamma)] \in \{\gamma, \delta\}$.

2.3. Linguistic time series-based forecasting model

As presented above, HA provides a mathematical formalism to generate a connection bridge between the computational objects and the qualitative word semantics. Based on HA, Hieu et al. introduced a new concept of LTS, which is applied to establish LTS-FM. After that, LTS-FM is applied to deal with the enrollment problem [25], in which the numeric TS is transformed into LTS using the numeric semantics (quantitative semantics) of the words selected to partition the numeric TS. Therefore, each datum of a given TS, whether stored as a numeric or linguistic value, can be naturally linked to a real numerical semantic of the corresponding word used in LTS. This is also the outstanding and different property of LTS compared to FTS. To provide a theoretical background for our proposed forecasting model, we briefly present some LTS concepts proposed in [25] as follows.

Definition 3: [25] Let $\mathcal{W}$ be a word set of an LV $\mathcal{N}$, and U be the UoD of $\mathcal{N}$. Then, any series $N(k)$, k = 0, 1, 2, ..., where $N(k)$ is a word collection taken from $\mathcal{W}$, is called a linguistic time series (LTS).

In Definition 3, k is a temporal variable. Real-world data collected in chronological order will generate an LTS. For example, when observing the closing price of a stock on the stock market over a period, we might obtain an LTS like: ..., "*very low*", "*quite low*", "*very low*", "*less low*", "*low*", "*very high*", "*high*", "*less high*", "*quite high*", "*high*", ... Or when observing the weather and assessing the average temperature for each day, the LTS might appear as: ..., "*cool*", "*slightly cool*", "*cool*", "*slightly hot*", "*hot*", "*quite hot*", "*very very hot*", "*hot*", ... Usually, $N(k)$ is a finite set. For example, when referring to the weather on a given day, we typically use a certain number of words to describe the degree of hotness or coldness. In LTS, linguistic values tend to form natural relationships with each other. Like fuzzy logic relationships, semantic logical relationships (SLRs) can be defined in a formal method as follows:

Definition 4: [25] Assume two words o_i and o_j are used to describe two data points at two times k and $k+1$, respectively. There is a relationship between o_i and o_j, called the semantic logical relationship (SLR), denoted as $o_i \to o_j$.

In fact, when considering a specific dataset, we can observe many SLRs that have the same left-hand side (LHS). This means that a single word o_i can potentially generate multiple different forecasted values. Therefore, the SLRs with the same LHS are put into semantic logical relationship groups (SLRGs).

Definition 5: [25]

Suppose that we have the SLRs as follows:

$$o_i \to o_{j1}, o_i \to o_{j2}, \dots, o_i \to o_{jn}$$

Then, those SLRs are put into an SLRG as: $o_i \to o_{j1}, o_{j2}, \dots, o_{jn}$.

Definition 6. [36] Let o_k be a linguistic word in $\mathcal{W}$ at a given time k caused by o_{k-1}, o_{k-2}, ..., o_{k-m} at the times $k-1$, $k-2$, ..., $k-m$, respectively. The obtained SLR represented by $o_{k-m}, \dots, o_{k-2}, o_{k-1} \to o_k$ is so-called m^{th}order SLR.

We can see in Definition 6 that an SLR is the first-order when $m = 1$, and an SLR is the high-order when $m > 1$.

By utilizing the concept of SLRG presented in Definition 5 for the high-order SLRs, the high-order SLRGs are established. Assume that we have the following high-order SLRs:

$$\begin{gathered} o_{k-m}, \dots, o_{k-2}, o_{k-1} \to o_{k1}, \\ o_{k-m}, \dots, o_{k-2}, o_{k-1} \to o_{k2}, \\ \dots, \\ o_{k-m}, \dots, o_{k-2}, o_{k-1} \to o_{km}, \end{gathered}$$

they get together to make a high-order SLRG as follows:

$$o_{k-m}, \dots, o_{k-2}, o_{k-1} \to o_{k1}, o_{k2}, \dots, o_{km},$$

where m is the order of SLRG.

Based on the concepts described above, the first-order LTS-FM introduced in [25] is described in Fig. 1.

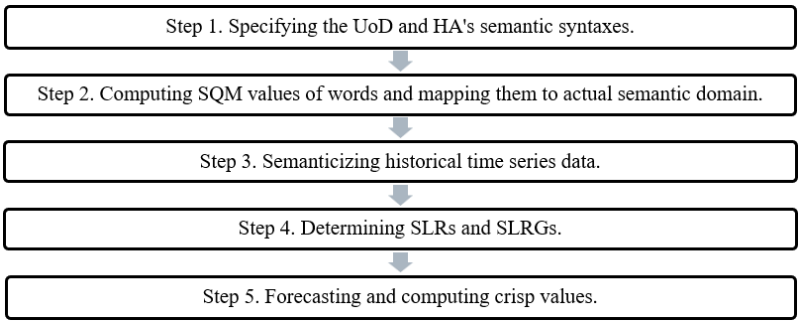


Fig. 1. The brief descriptions of the algorithm steps of the LTS-FM proposed in [25].

2.4. Particle Swarm Optimization

Given introduction in 1995 by Kennedy and Eberhart, Particle Swarm Optimization (PSO) [37] has been widely utilized in practice. PSO uses the following parameters to control its algorithm. Each particle x_i ($i = 1, \dots, N$) in a swarm has the position $\mathcal{X}_i^t$ at the loop t re-computed as follows:

$$\mathcal{X}_i^t = \mathcal{X}_i^{t-1} + \mathcal{V}_i^t, \tag{1}$$

and its velocity V_i^t is updated by its best position P_i^t and the global best position P_g^t in the swarm as follows:

$$\mathcal{V}_i^t = \varphi\mathcal{V}_i^{t-1} + \mathfrak{h}_1 n_1(P_i^t - \mathcal{X}_i^{t-1}) + \mathfrak{h}_2 n_2(P_g^t - \mathcal{X}_i^{t-1}), \quad (2)$$

where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two elements used to balance the self and social cognition of x_i; φ denotes the inertia ratio; n_1 and n_2 are two real numbers randomly generated in [0, 1]. Hereafter are the PSO's steps:

Step 1: Randomly initialize two vectors $\mathcal{V}$ and $\mathcal{X}$, the number of loops G_{max}, and the loop variable τ.
Step 2: Update the objective function value (OFV) $f(\mathcal{X}_i^t)$ and P_i^t of each particle x_i at the loop τ.
Step 3: Update P_g^t based on the best particle's position in the swarm.
Step 4: Specify $\mathcal{V}_i^t$ based on (2) and $\mathcal{X}_i^t$ based on (1).
Step 5: t = t + 1. If t ≥ $G_{\max}$ then terminate, else go to *Step* 2.

2.5. The forecasting model evaluation criteria

The first used model criterion is MSE (Mean Squared Error):

$$\text{MΣE} = \frac{1}{n}\sum_{i=1}^{n}(F_i - A_i)^2 \quad (3)$$

where n is the number of forecasting data values, A_{i} and F_{i} are the actual and forecasted data values, respectively.

The second used model criterion is MAPE (Mean absolute percentage error):

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{F_i - A_i}{F_i}\right| \times 100\%. \quad (4)$$

2.6. The proposed forecasting model

2.7. Two-factor SLR and two-factor SLRG

This subsection presents two concepts of the two-factor SLR and two-factor SLRG proposed in [38], which are used to design two-factor LTS-FMs. These concepts are able to be generalized to multi-factors.

Definition 7: [38] Suppose that there exist two LTSs L(k) and $S(k), k = \cdots, 0, 1, 2, \ldots$, where $L(k)$ and $S(k)$ are the LTSs of the main and secondary factors, respectively. Two words o_i and o_j describe the data at two given times k and $k + 1$ on *L(k)*, respectively. The word o_i describes the datum at the time k on *S(k)*. Then, a relationship exists between (o_i, v_i) and o_j, and this relationship is called two-factor SLR (SLR2F), denoted by:

$$(o_i, v_i) \rightarrow o_j$$

Definition 8: [38] Assume that we have the following SLR2Fs:

$$(o_i, v_i) \rightarrow o_{j1}, (o_i, v_i) \rightarrow o_{j2}, (o_i, v_i) \rightarrow o_{jn}.$$

Then, they are got together into a group of two-factor SLR (SLRG2F) as follows:

$$(o_i, v_i) \rightarrow o_{j1}, o_{j2}, \ldots, o_{jn}.$$

Like Definition 8, we can construct the high-order two-factor SLRG (HOSLRG2F) from the high-order SLR2Fs (HOSLR2Fs). Assume that we have the HOSLR2Fs as follows:

$$\begin{gathered}(o_{k-\lambda}, v_{k-\lambda}), \ldots, (o_{k-2}, v_{k-2}), (o_{k-1}, v_{k-1}) \rightarrow o_{k1} \\ (o_{k-\lambda}, v_{k-\lambda}), \ldots, (o_{k-2}, v_{k-2}), (o_{k-1}, v_{k-1}) \rightarrow o_{k2}, \\ \ldots, \\ (o_{k-\lambda}, v_{k-\lambda}), \ldots, (o_{k-2}, v_{k-2}), (o_{k-1}, v_{k-1}) \rightarrow o_{km},\end{gathered}$$

they get together to make an HOSLRG2F as follows:

$$(o_{k-\lambda}, v_{k-\lambda}), \dots, (o_{k-2}, v_{k-2}), (o_{k-1}, v_{k-1}) \rightarrow o_{k1}, o_{k2}, \dots, o_{km}.$$

2.8. The time-dependent SLRG and multi-factor time-dependent SLRG

As we can see in Definitions 5 and 8, all SLRs (or SLR2Fs) having the same LHS are put into an SLRG (or SLRG2F) regardless of when they appeared. Therefore, an SLR (or SLR2F) that occurred in the future can be used to forecast the past forecasted values in the training phase. This is not reasonable. In [31], the concept of time-dependent SLRG was proposed, so-called linguistic forecasting rules, in such a way that only the SLRs that occurred no later than the current time t are grouped into a time-dependent SLRG.

For example, assume that there are the SLRs: *Less few* → *Less few* and *Less few* → *medium* appeared in the years 1987 and 1988, respectively. The time-dependent SLRGs established for those years are *Less few* → *Less few* and *Less few* → *Less few*, *medium,* respectively. It can be seen that these SLRs have the same LHS, but the SLR *Less few* → *medium* that occurred in 1988 is not put into the same group as the SLRG established in 1987 because that SLR occurred later than that year.

In this paper, we extend the time-dependent SLRG to design the multi-factor time-dependent SLRGs.

Definition 10: Assume there exist the following SLR2Fs that occurred at times $k, k1, k2, \dots, kp$ with $k1, k2, \dots, kp \leq k$:

$$\begin{gathered}(o_i, v_i)(k-1) \longrightarrow o_{j1}(k) \\ (o_i, v_i)(k1-1) \longrightarrow o_{j2}(k1), \\ \dots, \\ (o_i, v_i)(kp-1) \longrightarrow o_{jn}(kp)\end{gathered}$$

They get together to make a two-factor time-dependent SLR group (SLRG_2FTD) as follows:

$$(o_i, v_i)(k-1) \longrightarrow o_{j1}(k), o_{j2}(k1), \dots, o_{jn}(kp)$$

2.9. The proposed two-factor time-dependent LTS forecasting model

In this section, a first-order two-factor time-dependent LTS forecasting model (LTS2FTD-FM) is proposed and presented using the concept of two-factor time-dependent SLRG in Fig. 2.

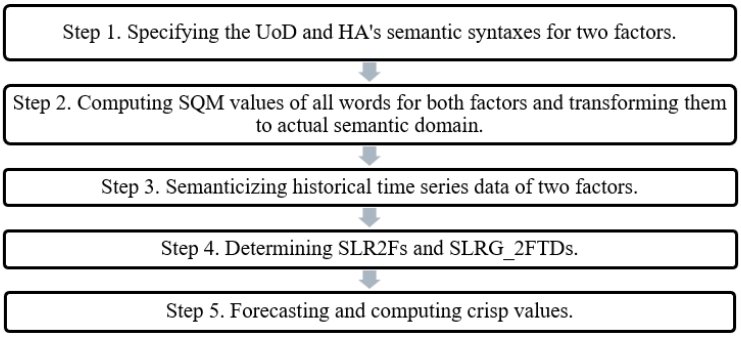


Fig. 2. Brief description of the algorithm steps of the proposed LTS2FTD-FM.

The fuzzy forecasted values in Step 5 are induced using SLRG_2FTDs, and the crisp forecasted values are specified as follows:

- At a certain time, say k, if the main factor's word is o_k and the secondary factor's word is o_k, and there is an SLRG_2FTD of the form $(o_k, v_k) \rightarrow o_{l1}, o_{l2}, \dots, o_{lp}$, p ≥ 1, the fuzzy value forecasted at the next time $k+1$ is $o_{l1}, o_{l2}, \dots, o_{lp}$. Then, this fuzzy value is defuzzified into a crisp value by $(\vartheta_r + \frac{\vartheta_{l1}+\vartheta_{l2}+\dots+\vartheta_{lp}}{p})/2$, where $\vartheta_{l1}, \vartheta_{l2}, \dots, \vartheta_{lp}$ are the numerical semantic values of $o_{l1}, o_{l2}, \dots, o_{lp}$, respectively; and ϑ_r is the numerical semantic value of o_k.

- At a certain time, say k, if the main factor's word is o_k and the secondary factor's word is v_k; and there is an SLRG_2FTD with an empty RHS (no word), the fuzzy value forecasted at the next time $k+1$ is o_k and it is defuzzied into ϑ_k, where ϑ_k is the numerical semantic value of o_k.

The first-order LTS2FTD-FM can be extended to the high-order LTS2FTD-FM by establishing high-order SLRG_2FTDs in Step 4 of the above forecasting procedure. Furthermore, the LTS2FTD-FM (both first-order and high-order) can be extended to forecasting models with more than two factors (multiple factors).

2.10. Co-optimize fuzzy parameter values and chosen wordset

The previous section presents our proposed two-factor time-dependent LTS forecasting model. However, the word collections chosen to fuzzify the historical TSs of both factors and the semantic parameter values (SPVs) of the associated HAs are intuitively chosen by experts. Therefore, they may not be good enough to produce good forecast results. This subsection presents a co-optimization algorithm for choosing linguistic words simultaneously with choosing SPVs based on PSO [37]. First, in the outer loop, a PSO is applied to choose the values of SPVs. Then, they are used as the input to an optimization algorithm for choosing the best linguistic word set in the inner loop. The algorithm output includes two sets: the SPVs and the optimal word set. They are the hyperparameter values optimized for the applied forecasting problems.

In all our experiments, only two LHs, Less (L) and Very (V) were applied. Two primary words were $\mathcal{p}^-$ and $\mathcal{p}^+$. Therefore, there are only four SPVs for both factors, including $m_M(\mathcal{p}^-), \mu_M(L), m_S(\mathcal{p}^-)$, and $\mu_S(L)$, that are the FMs of the negative primary words and the corresponding linguistic hedge L. The constraints on those values are as follows:

- For the main factor: $\mathfrak{w}_M(\mathcal{p}^-) + \mathfrak{w}_M(\mathcal{p}^+) = 1$ and $\mu_M(L) + \mu_M(V) = 1$, so we can infer that $\mathfrak{w}_M(\mathcal{p}^+) = 1 - \mathfrak{w}_M(\mathcal{p}^-)$ and $\mu_M(V) = 1 - \mu_M(L)$.
- For the secondary factor: $\mathfrak{w}_S(\mathcal{p}^-) + \mathfrak{w}_S(\mathcal{p}^+) = 1$ and $\mu_S(L) + \mu_S(V) = 1$, so we have $\mathfrak{w}_S(\mathcal{p}^+) = 1 - \mathfrak{w}_S(\mathcal{p}^-)$ and $\mu_S(V) = 1 - \mu_S(L)$.

A particle x_i in the swarm of the outer PSO represents a set of SPVs of both factors, that is $Z_i^t = \{\mathfrak{w}_M(\mathcal{p}^-), \mu_M(L), \mathfrak{w}_S(\mathcal{p}^-), \mu_S(L)\}, i = 1, \ldots, N$, where t is the loop step, and N is the number of particles in the swarm. Besides, a particle x_i in the swarm of the inner PSO represents a candidate solution of chosen word sets for two factors, where each word is coded by a real number in [0, 1], then it is transformed into an integer number representing the word index in the word domains. For the main factor, the word domain of $\mathcal{N}_L$ is represented by the vector $p_{L,i} = (w_1, \ldots, w_{d_w}, w_j \in [0,1])$. Therefore, the word set X_i^t chosen from the word domain W_{set} of $\mathcal{N}_L$ to describe the main factor is specified as follows:

$$X_i^t = \{x_l \in W_{set} | \, l = \lfloor w_j \times |W_{set}| \rfloor, 0 \le l < |W_{set}|\} \tag{5}$$

where, takes the integer part of a real number, |.| takes the cardinality of a set.

For the secondary factor, the word domain of $\mathcal{N}_S$ is represented by the vector $p_{S,i} = (s_1, \ldots, s_{d_s}, s_j \in [0,1])$ and word set Y_i^t chosen from the word domain S_{set} of $\mathcal{N}_S$ is specified as follows:

$$Y_i^t = \{y_r \in S_{set} | \, r = \lfloor s_j \times |S_{set}| \rfloor, 0 \le r < |S_{set}|\} \tag{6}$$

As the presentations of the particles of the outer and inner PSOs described above, the co-optimization algorithm of SPVs (TD_PSCO_SPVO) and word sets (TD_UWO) for two factors are presented in Fig. 3 and Fig. 4, respectively.

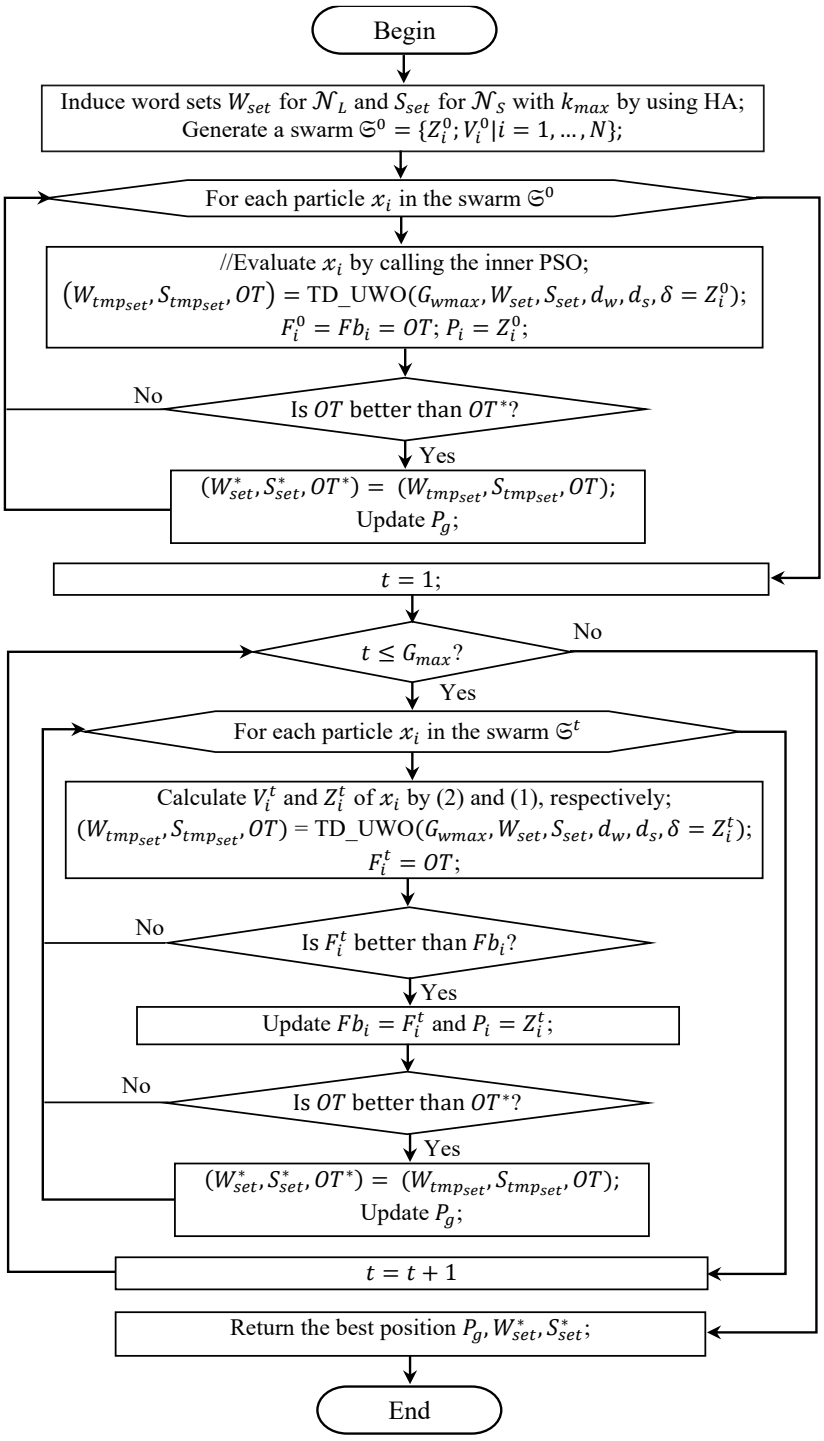


Fig. 3. The optimization algorithm of SPVs (TD_PSCO_SPVO) for two factors, the outer loop of the entire co-optimiztion algorithm.

The input of the algorithm TD_PSCO_SPVO is the dataset $\boldsymbol{D}$, HA's semantic syntaxes, and hyperparameter values of $N, G_{wmax}, G_{max}, k_{max}, d_w,$ and d_s, which are the number of loops of the inner and outer PSOs, the maximum length of words, the number of chosen words of both factors, respectively. The outputs are the sets of values of SPVs for two factors $Z^* = \{\mathfrak{w}_M(\mathscr{p}^-), \mu_M(L), \mathfrak{w}_S(\mathscr{p}^-), \mu_S(L)\}$, which are stored in the output variable P_g, and two optimal word sets W_{set}^* and S_{set}^* for both factors, respectively. Those hyperparameter value sets will be applied to the forecasting problems under consideration. The brief description of the algorithm body described in Fig. 3 is as follows:

- First, the initialization step is as follows: generate two word sets W_{set} for $\mathcal{N}_L$ and S_{set} for $\mathcal{N}_S$ with the length of k_{max} by using HA; Generate a swarm $\mathfrak{S}^0 = \{Z_i^0; V_i^0 | i = 1, \dots, N\}$, where $Z_i^0 = \{\mathfrak{w}_M(\mathcal{P}^-), \mu_M(L), \mathfrak{w}_S(\mathcal{P}^-), \mu_S(L)\}$ is the position and V_i^0 is the velocity of the particle x_i; $(W_{set}^*, S_{set}^*, OT^*) = (\emptyset, \emptyset, +\infty)$, where W_{set}^*, S_{set}^*, and OT^* store the global best chosen word sets for both factors and the global best value of the objective function, respectively.
- Second, evaluate the initial swarm:

  For each particle x_i in $\mathfrak{S}^0$,
  + Call the inner PSO namely TD_UWO($G_{wmax}, W_{set}, S_{set}, d_w, d_s, \delta = Z_i^0$) described in Fig. 4, whose returned values are $W_{tmp_{set}}, S_{tmp_{set}}$, and OT;
  + Update x_i's best position $P_i = Z_i^0$ and its current OFV $F_i^0 = OT$ and its local best OFV $Fb_i = OT$;
  + If OT is better than OT^*, update $(W_{set}^*, S_{set}^*, OT^*) = (W_{tmp_{set}}, S_{tmp_{set}}, OT)$ and the global best position of the swarm $P_g = P_i$;
- Third, repeat the loops from $t = 1$ until $t > G_{max}$:

  For each particle x_i in $\mathfrak{S}^t$,
  + Update its velocity V_i^t and its position Z_i^t using (2) and (1), respectively;
  + Evaluate the particle x_i by calling TD_UWO;
  + Update its best objective function $Fb_i = F_i^t$ and local best position $P_i = Z_i^t$ if F_i^t is better than Fb_i;
  + If OT is better than OT^*, update $(W_{set}^*, S_{set}^*, OT^*) = (W_{tmp_{set}}, S_{tmp_{set}}, OT)$ and $P_g = P_i$;

  t = t + 1;

The optimization algorithm of word sets, namely TD_UWO presented in Fig. 4, is the inner PSO of the outer PSO, namely TD_PSCO_SPVO presented in Fig. 3. Therefore, the input of TD_UWO is a list of parameter values of $G_{wmax}, W_{set}, S_{set}, d_w, d_s,$ and δ, which are the number of loops, the word sets of $\mathcal{N}_L$ and $\mathcal{N}_S$, the number of chosen words from W_{set} and S_{set}, and the set of SPVs chosen by the outer loop, respectively. The output of TD_UWO is the best chosen word sets for both factors saved in the variable $\mathcal{S}_g = \{W_{set}^*, S_{set}^*\}$ and *MAPE* or *MSE* value (depending on the specific forecasting problems). The algorithm chart in Fig. 4 is explained as follows:

- First, generate a random swarm $\mathfrak{S}^0 = \{X_i^0; V_i^0\} \cup \{Y_i^0; T_i^0\}, i = 1, \dots, M\}$, where X_i^0 and Y_i^0 are two subsets of indices of words in the word domains of the main and secondary factors, $X_i^0 \in [0, |W_{set}|]$ and $Y_i^0 \in [0, |S_{set}|]$; V_i^0 and T_i^0 are the velocities of X_i^0 and Y_i^0, respectively.
- Second, evaluate the initial swarm:

  For each particle x_i in $\mathfrak{S}^0$,
  + Call the proposed forecasting method LTS2FTD-FM with the inputs δ, X_i^0 and Y_i^0, where δ is the FPVs' set chosen by the outer PSO;
  + Compute the OFV F_i^0 by (3) or (4);
  + Update the personal best OFV $Fb_i = F_i^0$ and the local best position $\mathcal{S}_i = \{X_i^0, Y_i^0\}$;
- Third, repeat the loops from $t = 1$ until $t > G_{wmax}$:

  For each particle x_i in $\mathfrak{S}^t$,
  + Update its velocities V_i^t and T_i^t by (2), then update its positions $\mathcal{S}_i = \{X_i^t, Y_i^t\}$ by (1);
  + Sort the elements of X_i^t and Y_i^t to make sure they are in the right (increasing) order;
  + Call the proposed forecasting method LTS2FTD-FM with the inputs δ, X_i^t and Y_i^t;
  + Compute the OFV F_i^t by (3) or (4);
  + If F_i^t is better than Fb_i, update the personal best OFV $Fb_i = F_i^t$ and the local best position $\mathcal{S}_i = \{X_i^t, Y_i^t\}$.

  Update the global best position $\mathcal{S}_g$ based on the best value of Fb_i in the swarm t; $t = t + 1$;

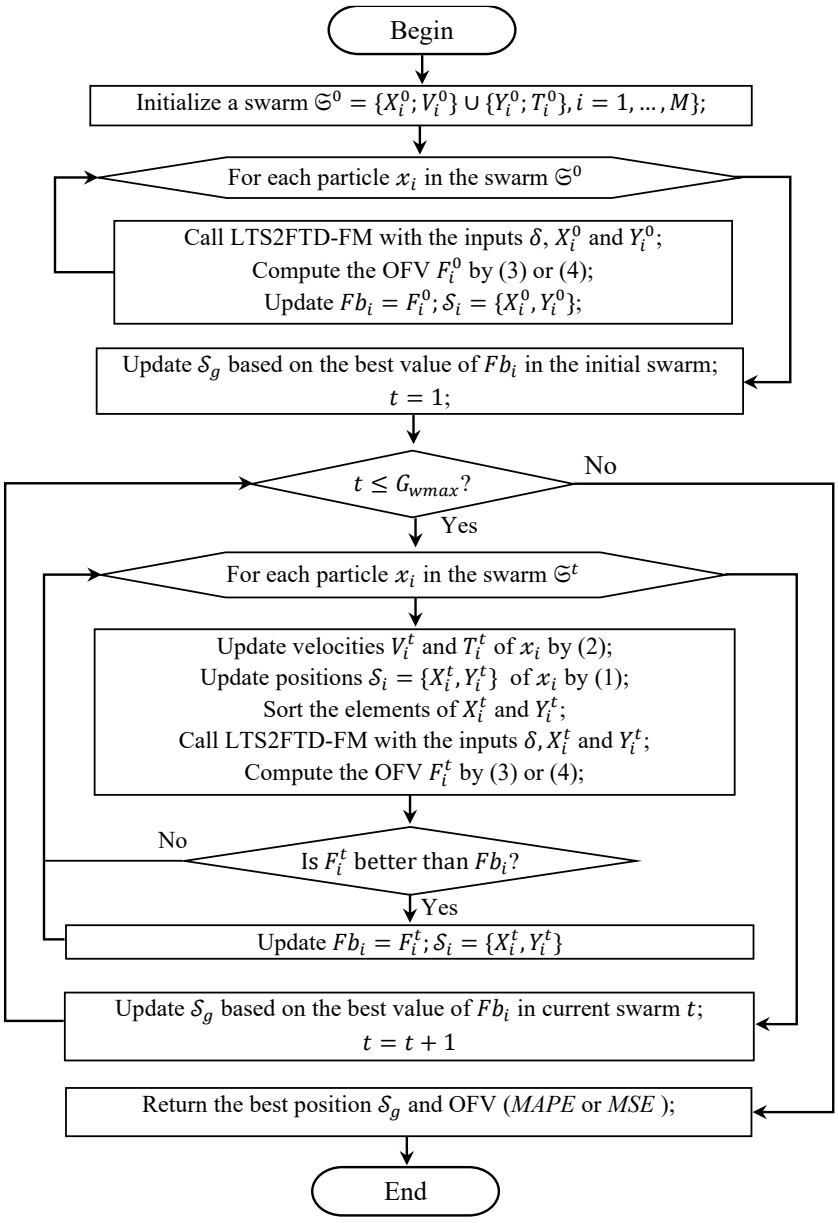


Fig. 4. The optimization algorithm of word sets (TD_UWO) for two factors, the inner loop of the entire co-optimization algorithm.

3. Results and Discussion

To demonstrate the efficiency of our proposed LTS2FTD-FMs (first- and high-order) with the published FTS-FMs and LTS-FMs, particularly two-factor models, this section presents the comparative assessments of the forecasted accuracies of the compared models.

3.1. The experiment setup

The proposed LTS2FTD-FMs are programmed using C# and executed on Windows 11 64-bit) on a computer with a Core i7 processor and 16GB RAM. The parameter values of PSO are set up as follows:

- For both inner and outer loops: inertia proportion is 0.4, $\mathfrak{h}_1 = \mathfrak{h}_2 = 1.5$.
- The number of iterations of the inner and outer PSOs are 20 and 100, respectively.
- The number of particles of the inner and outer PSOs are 30 and 20, respectively.

Each experimental model is executed three times on a historical dataset. Therefore, three MAPE and three MSE values are obtained, and the smallest ones are taken as the experimental results.

3.2. Forecast TAIFEX of Taiwan

Our first experiment focuses on forecasting the TAIFEX index values from Aug 3rd, 1998, to Sep 30th, 1998, with the TAIFEX and TAIEX indexes acting as the main and secondary factors, respectively.

Table 1. Comparison of the forecasting results of the proposed LTS2FTD-FM based on 7th-order with those of the existing models.

Date	Actual TAIFEX	Chen1996	H2001a	Lee2006	Lee2008	MTPSO	LTS2022	LTS2F	Proposed
1998-08-03	7552								
1998-08-04	7560	7450	7450						
1998-08-05	7487	7450	7450						
1998-08-06	7462	7500	7450	7450					
1998-08-07	7515	7500	7500	7550					
1998-08-10	7365	7450	7450	7350					
1998-08-11	7360	7300	7350	7350					
1998-08-12	7330	7300	7300	7350	7329	7325.28	7331.47	7314.59	7327.56
1998-08-13	7291	7300	7350	7250	7289.5	7287.48	7284.58	7275.5	7301.03
1998-08-14	7320	7183.33	7100	7350	7329	7325.28	7331.47	7314.59	7327.56
1998-08-15	7300	7300	7350	7350	7289.5	7287.48	7331.47	7314.59	7301.03
1998-08-17	7219	7300	7300	7250	7215	7221.26	7223.1	7227.46	7221.4
1998-08-18	7220	7183.33	7100	7250	7215	7221.26	7223.1	7227.46	7221.4
1998-08-19	7285	7183.33	7300	7250	7289.5	7287.48	7284.59	7275.5	7275.25
1998-08-20	7274	7183.33	7100	7250	7289.5	7287.48	7284.59	7275.5	7275.25
1998-08-21	7225	7183.33	7100	7250	7215	7221.26	7223.1	7227.46	7221.4
1998-08-24	6955	7183.33	7100	6950	6949.5	6952.02	6943.57	6951.55	6944.24
1998-08-25	6949	6850	6850	6950	6949.5	6952.02	6943.57	6951.55	6944.24
1998-08-26	6790	6850	6850	6750	6796	6781.01	6781.94	6802.04	6793.27
1998-08-27	6835	6775	6650	6850	6848	6842.05	6844.36	6841.14	6838.07
1998-08-28	6695	6850	6750	6650	6698.5	6696.17	6709.2	6686.92	6699.7
1998-08-29	6728	6750	6750	6750	6726	6726.5	6709.2	6726.02	6723.77
1998-08-31	6566	6775	6650	6550	6569.5	6580.45	6570.63	6565.22	6557.29
1998-09-01	6409	6450	6450	6450	6417	6409.24	6424.35	6406.84	6418.92
1998-09-02	6430	6450	6550	6450	6417	6409.24	6424.35	6406.84	6418.92
1998-09-03	6200	6450	6350	6250	6205	6213.94	6207.69	6192.08	6211.05
1998-09-04	6403.2	6450	6450	6450	6417	6409.24	6424.35	6406.84	6418.92
1998-09-05	6697.5	6450	6550	6650	6698.5	6696.17	6709.2	6686.92	6699.7
1998-09-07	6722.3	6750	6750	6750	6726	6726.5	6709.2	6726.02	6723.77
1998-09-08	6859.4	6775	6850	6850	6848	6864.96	6860.42	6869.13	6863.85
1998-09-09	6769.6	6850	6750	6750	6763	6781.01	6781.94	6774.05	6769.88
1998-09-10	6709.75	6775	6650	6750	6726	6696.17	6709.2	6726.02	6699.7
1998-09-11	6726.5	6775	6850	6750	6726	6726.5	6709.2	6726.02	6723.77
1998-09-14	6774.55	6775	6850	6817	6763	6781.01	6781.94	6774.05	6769.88
1998-09-15	6762	6775	6650	6817	6763	6781.01	6781.94	6774.05	6769.88
1998-09-16	6952.75	6775	6850	6817	6949.5	6952.02	6943.57	6951.55	6944.24
1998-09-17	6906	6850	6950	6950	6904.5	6906.7	6901.0	6903.52	6890.39
1998-09-18	6842	6850	6850	6850	6848	6842.05	6844.36	6841.14	6838.07
1998-09-19	7039	6850	6950	7050	7064	7039.2	7030.21	7038.67	7026.97
1998-09-21	6861	6850	6850	6850	6848	6864.96	6860.42	6869.13	6863.85
1998-09-22	6926	6850	6950	6950	6904.5	6906.7	6943.57	6923.55	6944.24
1998-09-23	6852	6850	6850	6850	6848	6842.05	6844.36	6841.14	6863.85
1998-09-24	6890	6850	6950	6850	6904.5	6906.7	6901.0	6903.52	6890.39
1998-09-25	6871	6850	6850	6850	6848	6864.96	6860.42	6869.13	6863.85
1998-09-28	6840	6850	6750	6850	6848	6842.05	6844.36	6841.14	6838.07
1998-09-29	6806	6850	6750	6850	6796	6781.01	6781.94	6802.04	6793.27
1998-09-30	6787	6850	6750	6750	6796	6781.01	6781.94	6774.05	6793.27
MSE		9668.94	7856.5	1364.56	105.02	92.17	140.01	76.34	64.5

To generate a rich word set for selecting the best finite word set to describe historical TS, the value of *k*, which determines the longest length of words, was set to 5. The cardinality of the finite word set applied to fuzzify the TS data of the Taiwan TAIFEX index for the main and secondary factors is 16, two primary words used are $\mathcal{P}^-$= *"low"* and $\mathcal{P}^+$= *"high"*, two used LHs are $L \in \mathcal{G}^-$ and $V \in \mathcal{G}^+$. The UoD of the main and secondary factors are U_L= [6100, 7600] and U_S= [6200, 7610], respectively.

To exhibit the efficiency of our proposed LTS2FTD-FMs over the published models, the forecasted data and accuracy of LTS2FTD-FMs were compared with those of Chen in [2] (Chen1996), Huarng in [39] (H2001a), Lee in [11] (Lee2006), Lee in [40] (Lee2008), the MTPSO model in [12], the single-factor LTS-FM (LTS2022) in [41] and the two-factor time-independent LTS-FM (LTS2F) in [38]. The average *MSE* values are used for evaluating the accuracy of forecasting models, and their comparison results are visualized in Fig. 3 and indicated in Table 1.

As shown in Table 1 and Fig. 5, our proposed LTS2FTD-FM with the integration of 7^{th}-order time-dependent SLRG2Fs has an MSE value of 64.5, which is the best value in comparison with the existing models such as Chen1996, H2001a, Lee2006, Lee2008, MTPSO, LTS2022, and LTS2F. Especially, the LTS2F model, which applied the time-independent SLRG2F, has an MSE value of 76.34, which is much worse than our proposed model. Furthermore, as seen in Fig. 3, the curve representing the forecasted data of LTS2FTD-FM is closest to the one representing the actual data. Therefore, it can be stated that the time-dependent SLRG2F is the main cause that improves the forecast accuracy.

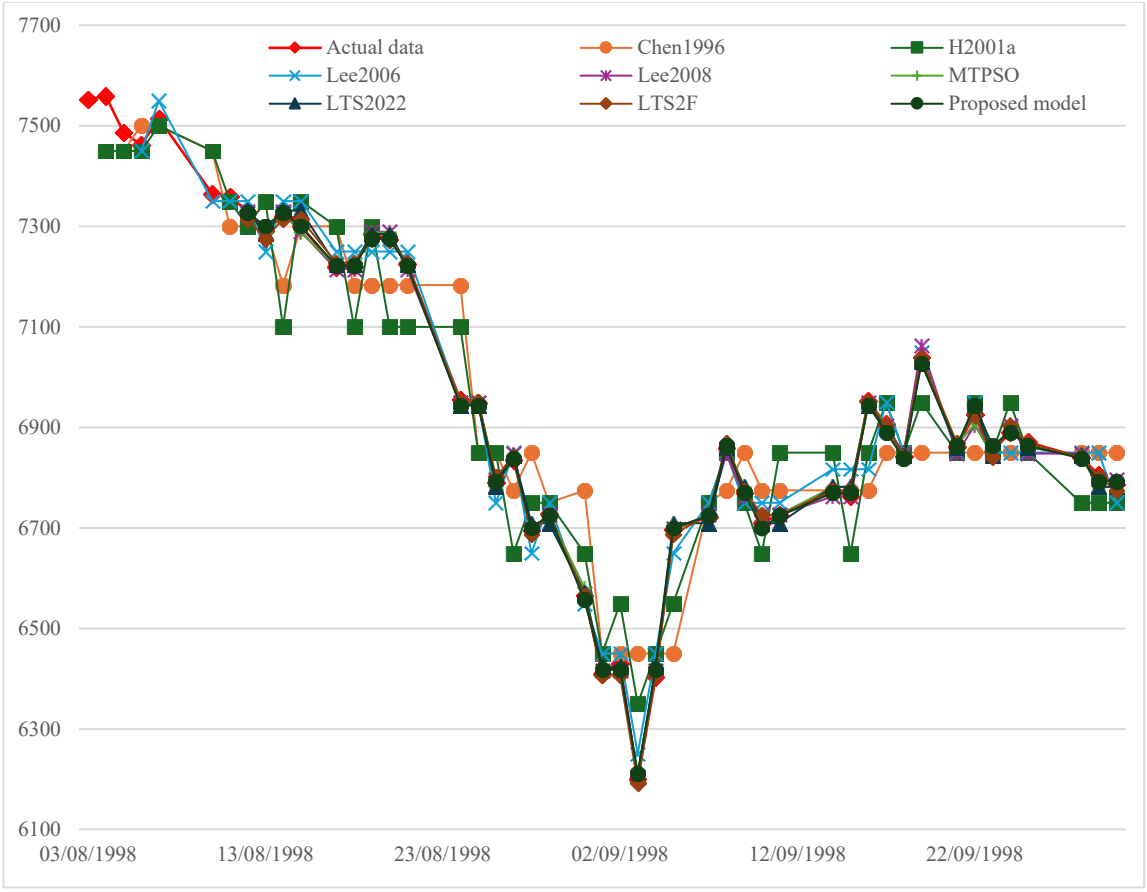


Fig. 5. The comparison chart of the proposed LTS2FTD-FM based on 7^{th}-order with the existing forecasting models.

3.3. Forecast temperature in Taipei

To have a diversity comparison to show the significant improvement of our proposed model, we conducted the second experiment on the forecasting problem of daily average temperature in Taipei.

Two data attributes contribute to the forecasting process in such a way that the Temperature and Cloud Density attributes act as the main and secondary factors, respectively.

The HA's semantic syntaxes were set up as follows:

- The value of k, which determines the maximum word length, was set to 4.
- As in the previous studies, the cardinalities of the chosen word sets for the main and secondary factors are 9 and 7, respectively.
- Two chosen primary words are $p^- = "cold"\ and\ p^+ = "hot"$, respectively. Two LHs are also $L \in \mathcal{G}^-$ and $V \in \mathcal{G}^+$.

As suggested by human experts, the UoDs of the months from Jun. to Sep. are specified as follows: June: U_L = [26, 31] and U_S= [9, 97], July: U_L = [26, 32] and U_S = [7, 100], August: U_L = [25.5, 30.5] and U_S = [13, 100], September: U_L = [23, 31] and U_S = [2, 84].

Table 2. The comparison of forecasting results of the proposed LTS2FTD-FM versus those of the benchmarked models on the data in June 1996 in the case of the first and higher orders.

Model		Order of SLR							
		1	2	3	4	5	6	7	8
Chen2000		-	2.88%	3.16%	3.24%	3.33%	3.39%	3.53%	3.67%
Lee2007		1.24%	0.74%	0.64%	0.72%	0.65%	0.66%	0.64%	0.65%
	α=0.25	0.79%	0.44%	0.42%	0.42%	0.42%	0.44%	0.40%	0.40%
Lee2008	α =0.5	0.84%	0.50%	0.45%	0.42%	0.38%	0.43%	0.39%	0.46%
	α =0.9	0.79%	0.46%	0.42%	0.44%	0.42%	0.41%	0.46%	0.39%
MPSO		0.45%	0.36%	0.34%	0.32%	0.31%	0.31%	0.28%	0.29%
LTS2022		0.68%	0.26%	0.30%	0.27%	0.28%	0.28%	0.28%	0.26%
LTS2F		0.26%	0.23%	0.24%	0,21%	0,20%	0.19%	0.15%	0.17%
Proposed model		**0.24%**	**0.23%**	**0.23%**	**0.20%**	**0.19%**	**0.18%**	**0.14%**	**0.15%**

Table 3. The comparisons of forecasting results of proposed LTS2FTD-FM versus those of the benchmarked models on the data in July 1996 in case of the first and high-orders.

Model		Order of SLR							
		1	2	3	4	5	6	7	8
Chen2000		-	3.04%	3.76%	4.08%	4.17%	4.35%	4.38%	4.56%
Lee2007		1.23%	0.78%	0.73%	0.83%	0.70%	0.71%	0.68%	0.69%
	α=0.25	0.66%	0.45%	0.42%	0.41%	0.41%	0.40%	0.41%	0.40%
Lee2008	α =0.5	0.66%	0.50%	0.47%	0.44%	0.40%	0.38%	0.44%	0.42%
	α =0.9	0.62%	0.46%	0.45%	0.44%	0.44%	0.41%	0.40%	0.40%
MPSO		0.41%	0.34%	0.33%	0.33%	0.32%	0.32%	0.34%	0.33%
LTS2022		0.83%	0.46%	0.32%	0.33%	0.34%	0.32%	0.30%	0.30%
LTS2F		0.28%	0.27%	0.26%	0.26%	0.27%	0.28%	0.26%	0.26%
Proposed model		**0.26%**	**0.26%**	**0.26%**	**0.26%**	**0.26%**	**0.27%**	**0.25%**	**0.25%**

Table 4. Comparison of forecasting results of the proposed LTS2FTD-FM versus those of the benchmarked models on the data in August 1996 in the case of the first and higher orders.

Model		Order of SLR							
		1	2	3	4	5	6	7	8
Chen2000		-	2.75%	2.77%	3.30%	3.40%	3.18%	3.15%	3.19%
Lee2007		1.09%	0.92%	0.88%	1.07%	0.75%	0.76%	0.75%	0.73%
	α=0.25	0.64%	0.43%	0.47%	0.40%	0.41%	0.38%	0.40%	0.45%
Lee2008	α =0.5	0.69%	0.40%	0.38%	0.37%	0.37%	0.39%	0.42%	0.45%
	α =0.9	0.66%	0.40%	0.40%	0.40%	0.36%	0.41%	0.39%	0.44%
MPSO		0.41%	0.31%	0.34%	0.33%	0.33%	0.33%	0.34%	0.35%
LTS2022		0.74%	0.25%	0.28%	0.30%	0.30%	0.29%	0.29%	0.29%
LTS2F		0.25%	0.23%	0.22%	0.23%	0.22%	0.22%	0.23%	0.22%
Proposed model		**0.20%**	**0.22%**	**0.21%**	**0.22%**	**0.21%**	**0.22%**	**0.23%**	**0.22%**

Table 5. The comparison of forecasting results of the proposed LTS2FTD-FM versus those of the benchmarked models on the data in September 1996 in the case of the first and higher orders.

Model		Order of SLR							
		1	2	3	4	5	6	7	8
Chen2000		-	3.29%	3.10%	3.19%	3.22%	3.39%	3.38%	3.29%
Lee2007		1.28%	1.01%	0.90%	0.94%	0.96%	0.95%	0.95%	0.95%
Lee2008	α=0.25	0.69%	0.58%	0.59%	0.57%	0.56%	0.57%	0.58%	0.47%
	α =0.5	0.66%	0.62%	0.59%	0.59%	0.56%	0.54%	0.56%	0.53%
	α =0.9	0.62%	0.59%	0.61%	0.57%	0.54%	0.59%	0.57%	0.50%
MPSO		0.54%	0.54%	0.56%	0.54%	0.50%	0.51%	0.52%	0.41%
LTS2022		0.97%	0.41%	0.61%	0.59%	0.59%	0.61%	0.61%	0.63%
LTS2F		0.40%	0.34%	0.46%	0.40%	0.40%	0.41%	0.40%	0.33%
Proposed model		**0.36%**	**0.40%**	**0.43%**	**0.39%**	**0.39%**	**0.40%**	**0.40%**	**0.32%**

The results of the experiments of our proposed models are compared with those of the benchmarked models such as Chen [42] (Chen2000), Lee [3] (Lee2007), Lee [40] (Lee2008), MTPSO [12], Phong [41] (LTS2022), and the two-factor time-independent LTS-FM (LTS2F) in [38]. The forecasted accuracies of the proposed and benchmarked forecasting models, evaluated by MAPE values according to each month and the model orders, are exhibited in Tables 2, 3, 4, and 5. As seen in those tables, the MAPE values of the proposed models from the first-order to the eighth-order are mostly better than those of the benchmarked models (Chen2000, Lee2007, Lee2008, MPSO, LTS2022, and LTS2F), especially the first-order. Specifically, the MAPE values of the proposed first-order model on data in June, July, August, and September are 0.24%, 0.26%, 0.20%, and 0.36%, respectively, which are smaller (better) than those of the best benchmarked models, in turn, 0.02%, 0.02%, 0.05%, and 0.04%. The MAPE values of the high-order models of the proposed models are slightly better than those of the LTS2F model and are much better than those of the rest of the benchmarked models. It is easy to see that LTS2022 is also an LTS-FM, but its results are not as good because it is a single-factor model. It is based only on the historical temperature data for forecasting, while the two-factor models use the cloud density factor for additional information. Besides, both LTS2F and the proposed models exploit cloud density as the secondary factor to improve forecast accuracy. The experimental results show that the time-dependent high-order models are much better than the time-independent ones. These results may come from the data features of this forecasting problem; the time-dependent SLRGs add little new information. Therefore, this phenomenon is taken into account in our future research.

4. Conclusion

The forecasting models based on FTS have been widely studied and applied to solve the various real-world forecasting problems due to their advantages in handling uncertain data and having good enough forecasting accuracy. The forecasting models based on LTS apply an advanced methodology to automatically transform numerical TS into LTS and directly infer from LTSs by utilizing HA theory. The existing studies have shown that multi-factor LTS allow for the design of more efficient forecasting models than the single-factor ones. However, the current studies focus on the time-independent multi-factor SLRGs, where the multi-factor SLRs that occur in the future participate in forecasting the past forecasted values in the training phase. This sounds illogical for solving a machine learning problem. Furthermore, the time-dependent models allow the SLRGs to change over time, so they can adapt to the evolving patterns. In this paper, we propose the concept of a time-dependent two-factor SLRG in such a way that only the two-factor SLRs that occurred no later than the forecasting time are put into the RHS of a time-dependent two-factor SLRG, and we apply them to design the proposed LTS2FTD-FM. The experimental results on the problem of forecasting the TAIFEX index of Taiwan and daily average temperatures in Taipei (Taiwan) show that the proposed forecasting model achieves better forecast accuracy than the compared models with both first-order and higher orders. However, in spite of forecasted accuracy improvements, the proposed LTS2FTD-FM still has some limitations. The co-optimization algorithm applied to the proposed models consumes a large amount of running time, so it needs further refinement. Besides, the formula for computing crisp forecasted values is still simple; more

related information needs to be integrated into it to increase the forecasted accuracy. Future research directions focus on the refinement of the optimization algorithm and the enhancement of the crisp value computation. Besides, the concept of the time-dependent two-factor SLRG will be generalized to time-dependent multi-factor SLRG (three factors and higher) and applied to solve a great deal of practical multi-factor forecasting problems.

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